Harmonic function:

A real valued function u of two variables x and y is said to be harmonic if it has a continuous partial derivatives and satisfies the equation-

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow \nabla^2 = 0$$

This equation is also known as Laplace equation.

The functions u(x,y) and v(x,y) which satisfy Laplace's equation are called harmonic functions.

If $u(x, y) = x^2 - y^2$ and $v(x, y) = -\frac{y}{x^2+y^2}$ then show that both u and v satisfies Laplace's equation but f(z)=u+iv is not an analytic function of z.
Proof:-To show that, f(z)= u+iv is not an analytic function of z,we have to show that u and v do not satisfy the Cauchy-Riemann equations- $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ **and** $\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$ Now $\frac{\delta u}{\delta x} = 2x \, , \frac{\delta u}{\delta y} = -2y$ $\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}\left(-\frac{y}{x^2+y^2}\right) = -y\left[\frac{(-1)2x}{(x^2+y^2)^2}\right] = \frac{2xy}{(x^2+y^2)^2}$ $\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}\left(-\frac{y}{x^2+y^2}\right) = -y\left[\frac{-(x^2+y^2).1-(-y)2y}{(x^2+y^2)^2}\right] = -\frac{(x^2-y^2)}{(x^2+y^2)^2}$ Clearly $\frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$ **and** $\frac{\delta v}{\delta x} \neq -\frac{\delta u}{\delta y}$ $\Rightarrow$ u + iv is not an analytic function of z. To prove that u and v both satisfy Laplace's equation, we have to show that- $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \Rightarrow u_{xx} + u_{yy} = 0$ **and** $\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0 \Rightarrow v_{xx} + v_{yy} = 0$ **Now we have to find** $\frac{\delta^2 u}{\delta x^2}, \frac{\delta^2 u}{\delta y^2}, \frac{\delta^2 v}{\delta x^2}, \frac{\delta^2 v}{\delta y^2}$

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta x}(2x) = 2 \text{ and } \frac{\delta^2 u}{\delta y^2} = \frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta y}(-2y) = -2$$

$$\frac{\delta^2 v}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta v}{\delta x}\right)$$

$$= \frac{\delta}{\delta x}\left(\frac{2xy}{(x^2+y^2)^2}\right) = 2y\left[\frac{(x^2+y^2)^2 .1 - x.2(x^2+y^2)2x}{(x^2+y^2)^4}\right] = 2y\left[(x^2+y^2)\frac{(x^2+y^2)-4x^2}{(x^2+y^2)^4}\right] = 2y\left[\frac{y^2-3x^2}{(x^2+y^2)^3}\right]$$

$$\frac{\delta^2 v}{\delta y^2} = \frac{\delta}{\delta y}\left(\frac{\delta v}{\delta y}\right) = \frac{\delta}{\delta y}\left(-\frac{(x^2-y^2)}{(x^2+y^2)^2}\right)$$

$$= -\frac{\delta}{\delta y}\left(\frac{(x^2-y^2)}{(x^2+y^2)^2}\right) = -\left[\frac{(x^2+y^2)^2(-2y)-(x^2-y^2)2(x^2+y^2)2y}{(x^2+y^2)^4}\right] = -\left[(x^2+y^2)\ \frac{(x^2+y^2)(-2y)-(x^2-y^2)4y}{(x^2+y^2)^4}\right]$$

$$= -(-2y)\left[\frac{(x^2+y^2)+(x^2-y^2)2}{(x^2+y^2)^3}\right] = 2y\left[\frac{(3x^2-y^2)}{(x^2+y^2)^3}\right]$$

Thus

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 2 - 2 = 0 \text{ and } \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 2y\left[\frac{y^2-3x^2}{(x^2+y^2)^3}\right] + 2y\left[\frac{(3x^2-y^2)}{(x^2+y^2)^3}\right] = 0$$

- Prove that, $u(x,y) = e^{-x}(x\sin y - y\cos y)$ is harmonic. Find v such that $f(z) = u + iv$ is regular.

1st part:

$$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x}[e^{-x}(x\sin y - y\cos y)] = -e^{-x}(x\sin y - y\cos y) + e^{-x}(\sin y - 0)$$

$$= e^{-x}(-x\sin y + y\cos y + \sin y)$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta x}[e^{-x}(-x\sin y + y\cos y + \sin y)]$$

$$= -e^{-x}(-x\sin y + y\cos y + \sin y) + e^{-x}(-\sin y - 0 - 0) = e^{-x}(x\sin y - y\cos y - 2\sin y)$$

$$\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}[e^{-x}(x\sin y - y\cos y)] = e^{-x}[x\cos y - (\cos y - y\sin y)]$$

$$= e^{-x}[x\cos y - \cos y + y\sin y]$$

$$\frac{\delta^2 u}{\delta y^2} = \frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta y} e^{-x}[x\cos y - \cos y + y\sin y]$$

$$= e^{-x}[-x\sin y + \sin y + (\sin y + y\cos y)]$$

$$= e^{-x}[-x\sin y + 2\sin y + y\cos y]$$

Thus

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = e^{-x}[x\cos y - \cos y + y\sin y] + e^{-x}[-x\sin y + 2\sin y + y\cos y] = 0$$

2nd part: From Cauchy-Riemann equations we have

$$\frac{\delta v}{\delta y} = \frac{\delta u}{\delta x} \Rightarrow \frac{\delta v}{\delta y} = e^{-x}(-x\sin y + y\cos y + \sin y)$$

Integrate this with respect to y we get

$$v = \int e^{-x}(-x\sin y + y\cos y + \sin y)dy$$

$$= e^{-x}\int\{(-x+1)\sin y + y\cos y\}dy$$

$$= e^{-x}[\int(-x+1)\sin y\, dy + \int(y\cos y)dy]$$

$$= e^{-x}[\int(-x+1)\sin y\, dy + \int(y\cos y)dy]$$

$$= e^{-x}\left[(-x+1)(-\cos y) + y\int(\cos y)dy - \int\left\{\left(\frac{d}{dy}(y)\int \cos y\, dy\right)\right\}dy\right]$$

$$= e^{-x}[(-x+1)(-\cos y) + y\sin y - \int \sin y\, dy]$$

$$= e^{-x}[(-x+1)(-\cos y) + y\sin y + \cos y] + c$$

$$= e^{-x}[x\cos y - \cos y + y\sin y + \cos y] + c$$

$$= e^{-x}[x\cos y + y\sin y] + c$$

❖ **Self exercise**

(i) Prove that, $u(x,y) = \log\sqrt{x^2+y^2}$ **is harmonic. Find v such that** $f(z) = u + iv$ is regular.

(ii) Prove that, $u(\mathbf{x}, \mathbf{y}) = \mathbf{y}^3 - 3\mathbf{x}^2\mathbf{y}$ **is harmonic. Find v such that** $f(z) = u + iv$ is regular.